Approximate Solution of the Finite Cylinder Problem Using Legendre Polynomials

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Governing equations of elasticity for small deformations of a hollow cylinder of finite length are examined to define suitable approximate theories. Each dependent variable in the problem is represented as a series expansion in Legendre polynomials in the radial coordinate; such representations permit the field equations to be exactly reduced to two-dimensional sets of equations by separation of variables. Attention is directed toward establishment of a logical approach to truncation of the series; important variables for approximate theories of any order are established by consideration of the strain energy expression, and a truncation scheme is postulated. To establish the validity of the truncation scheme and hence of the resulting approximate theories, the problem of an infinitely long thick cylinder under axisymmetric band loading is considered. Comparisons with exact solutions and with well-known shell theories are obtained; results indicate that the new approximate theories yield predictions in better agreement with exact solutions than previous efforts in this area.

Nomenclature

$\bar{r},\theta,\bar{z}$	= cylindrical coordinates
η,z	= dimensionless coordinates
$\bar{\sigma}_{ij},\bar{u},\bar{v},\bar{w}$	= actual stresses and displacements
σ_{ij},u,v,w	= dimensionless stresses and displacements
h,R	= cylinder half-thickness and midsurface radius of curvature
$P_n(\eta),Q_n(\eta)$	= Legendre polynomials
ϵ	= dimensionless geometric parameter
$\sigma_{ij}^{(n)},S_{ij}^{(n)}$, etc.	= coefficient functions in Legendre series
N	= number defining order of approximate theory
E,ν	= isotropic elastic material constants

Introduction

THIS work reports on initial efforts to obtain approximate theories for analysis of thick-walled elastic shell structures under highly localized loadings. Attention is focused here only on cylindrical geometry but the method of approach is easily generalized to other shell configurations.

Solutions for finite length axisymmetric thick-walled isotropic elastic cylinders do exist in principle; a general analysis of this problem, along with references to related work, is presented in Ref. 1. Computational difficulties associated with these solutions are great, and they are not in a form readily used by practicing engineers for general mixed boundary conditions on end faces. The extension of the solution procedure to transversely isotropic cylinders and the presentation of additional results for isotropic cylinders are provided by Refs. 2 and 3; although the solution procedure generates a set of eigenfunctions satisfying the governing field equations and stress free conditions on the cylindrical surface, the lack of orthogonality relations makes difficult simultaneous satisfaction of two boundary conditions on the end faces of the cylinder.

The solution techniques presented in Refs. 1–3 satisfy end face boundary conditions in a least squares sense; completeness of the set of eigenfunctions cannot be rigorously proved but can only be inferred from numerical results of particular problems.

In this work, we present an alternate approach to the generation of exact and approximate solutions to the finite cylinder problem. Recognizing difficulties encountered in previous efforts, we assume series solutions in terms of the orthogonal, complete set of Legendre polynomials. Expansions of this form allow reduction of the three-dimensional elasticity equations to a coupled infinity of two-dimensional field equations. Orthogonality permits simultaneous satisfaction of all boundary conditions on the end faces, within the context of a series representation, and completeness insures convergence to the exact solution as more terms are included.

This work may be considered as an alternate approach to generating an exact solution to the axisymmetrically loaded finite cylinder; our motivation, however, is to gain insight into possible approaches to development of higher-order shell type theories that relate structural theories to the three-dimensional elasticity equations. Similar efforts with the plane stress problem have yielded approximate theories clearly illustrating the relationship between classical beam theory and higher-order theories capable of treating local effects.[4]

Derivation of the Governing Equations

Cylindrical shell geometry and nomenclature are indicated in Fig. 1. We introduce the following nondimensional forms for all stresses and deformations:

$$
\begin{aligned}
&u = \bar{u}/h \qquad v = \bar{v}/h \qquad w = \bar{w}/h \\
&z = \bar{z}/h \qquad \eta = (\bar{r} - R)/h \qquad \epsilon = h/R \\
&E\sigma_{rr} = (1 + \epsilon\eta)\bar{\sigma}_{rr} \qquad E\sigma_{\theta\theta} = (1 + \epsilon\eta)\bar{\sigma}_{\theta\theta} \\
&E\sigma_{zz} = (1 + \epsilon\eta)\bar{\sigma}_{zz} \qquad E\sigma_{r\theta} = (1 + \epsilon\eta)\bar{\sigma}_{r\theta} \\
&E\sigma_{rz} = (1 + \epsilon\eta)\bar{\sigma}_{rz} \qquad E\sigma_{\theta z} = (1 + \epsilon\eta)\bar{\sigma}_{\theta z}
\end{aligned} \tag{1}
$$

The field equations for the linear isotropic elasticity problem

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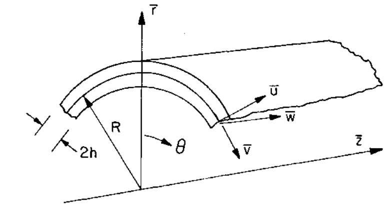

Fig. 1 Shell geometry and nomenclature.

then become:

$$(1+\epsilon\eta)\frac{\partial\sigma_{rr}}{\partial\eta}+\epsilon\frac{\partial\sigma_{r\theta}}{\partial\theta}+(1+\epsilon\eta)\frac{\partial\sigma_{rz}}{\partial z}-\epsilon\sigma_{\theta\theta}=0 \qquad (2a)$$

$$(1+\epsilon\eta)\frac{\partial\sigma_{r\theta}}{\partial\eta}+\epsilon\sigma_{r\theta}+\epsilon\frac{\partial\sigma_{\theta\theta}}{\partial\theta}+(1+\epsilon\eta)\frac{\partial\sigma_{\theta z}}{\partial z}=0 \qquad (2b)$$

$$(1+\epsilon\eta)\frac{\partial\sigma_{rz}}{\partial\eta}+\epsilon\frac{\partial\sigma_{\theta z}}{\partial\theta}+(1+\epsilon\eta)\frac{\partial\sigma_{zz}}{\partial z}=0 \qquad (2c)$$

$$(1+\epsilon\eta)\partial u/\partial\eta=\sigma_{rr}-\nu\sigma_{\theta\theta}-\nu\sigma_{zz} \qquad (2d)$$

$$(1+\epsilon\eta)\partial v/\partial z+\epsilon\partial w/\partial\theta=2(1+\nu)\sigma_{\theta z} \qquad (2e)$$

$$(1+\epsilon\eta)\partial w/\partial\eta+(1+\epsilon\eta)\partial u/\partial z=2(1+\nu)\sigma_{rz} \qquad (2f)$$

$$\epsilon\partial v/\partial\theta+\epsilon u=\sigma_{\theta\theta}-\nu\sigma_{rr}-\nu\sigma_{zz} \qquad (2g)$$

$$(1+\epsilon\eta)\partial w/\partial z=\sigma_{zz}-\nu\sigma_{rr}-\nu\sigma_{\theta\theta} \qquad (2h)$$

$$(1+\epsilon\eta)\partial v/\partial\eta-\epsilon v+\epsilon\partial u/\partial\theta=2(1+\nu)\sigma_{r} \qquad (2i)$$

We assume that all dependent variables may be represented in the form

$$\begin{Bmatrix}\sigma_{rr}\\ \sigma_{r\theta}\\ \sigma_{rz}\\ u\\ v\\ w\end{Bmatrix}=\sum_{n=0}^{\infty}\begin{Bmatrix}S_{rr}^{(n)}(\theta,z)\\ S_{r\theta}^{(n)}(\theta,z)\\ S_{rz}^{(n)}(\theta,z)\\ U^{(n)}(\theta,z)\\ V^{(n)}(\theta,z)\\ W^{(n)}(\theta,z)\end{Bmatrix}Q_n(\eta)=\sum_{n=0}^{\infty}\begin{Bmatrix}\sigma_{rr}^{(n)}(\theta,z)\\ \sigma_{r\theta}^{(n)}(\theta,z)\\ \sigma_{rz}^{(n)}(\theta,z)\\ u^{(n)}(\theta,z)\\ v^{(n)}(\theta,z)\\ w^{(n)}(\theta,z)\end{Bmatrix}P_n(\eta) \qquad (3a)$$

$$\begin{Bmatrix}\sigma_{\theta\theta}\\ \sigma_{zz}\\ \sigma_{\theta z}\end{Bmatrix}=\sum_{n=0}^{\infty}\begin{Bmatrix}\sigma_{\theta\theta}^{(n)}(\theta,z)\\ \sigma_{zz}^{(n)}(\theta,z)\\ \sigma_{\theta z}^{(n)}(\theta,z)\end{Bmatrix}P_n(\eta) \qquad (3b)$$

The polynomials $Q_n(\eta)$ are defined by

$$Q_n(\eta)=P_n(\eta)-P_{n-2}(\eta);\; P_n=0,\, n<0$$

Representations in terms of $Q_n(\eta)$ functions are particularly convenient as the first two terms describe completely behavior on surfaces $\eta=\pm1$. The two sets of coefficient functions are related by equations of the form

$$\sigma_{rr}^{(n)}=S_{rr}^{(n)}-S_{rr}^{(n+2)} \qquad n\geq 0 \qquad (4)$$

If we substitute Eqs. (3) into Eqs. (2), the governing field equations can be put into the form

$$\sum_{n=0}^{\infty}F_{in}(\theta,z)P_n(\eta)=0 \qquad i=1,2,\ldots 9 \qquad (5)$$

which require, for satisfaction for all η, that

$$F_{in}(\theta,z)=0 \qquad i=1,2,\ldots 9;\; n\geq 0 \qquad (5a)$$

Equations (5a) are a ninefold infinity of field equations defined in the two-dimensional space domain; they represent the starting point in the derivation of approximate shell type structural theories. Note that the use of a cylindrical geometry is not restrictive; the method of approach can be generalized to other geometries. In expanded form, Eqs. (5a) yield the typical forms ($n\geq 0$)

$$(2n+1)S_{rr}^{(n+1)}=-n\epsilon S_{rr}^{(n)}-(n+1)\epsilon S_{rr}^{(n+2)}-\epsilon\frac{\partial S_{r\theta}^{(n)}}{\partial\theta}+\epsilon\frac{\partial S_{r\theta}^{(n+2)}}{\partial\theta}-\frac{n\epsilon}{(2n-1)}\frac{\partial S_{rz}^{(n-1)}}{\partial z}-\frac{\partial S_{rz}^{(n)}}{\partial z}+\frac{(2n+1)\epsilon}{(2n-1)(2n+3)}\frac{\partial S_{rz}^{(n+1)}}{\partial z}+\frac{\partial S_{rz}^{(n+2)}}{\partial z}+\frac{(n+1)\epsilon}{(2n+3)}\frac{\partial S_{rz}^{(n+3)}}{\partial z}+\epsilon\sigma_{\theta\theta}^{(n)} \qquad (6a)$$

$$(2n+1)S_{r\theta}^{(n+1)}=\ldots \qquad (6b)$$

$$(2n+1)S_{rz}^{(n+1)}=\ldots \qquad (6c)$$

$$(2n+1)U^{(n+1)}=-n\epsilon U^{(n)}-(n+1)\epsilon U^{(n+2)}+S_{rr}^{(n)}-S_{rr}^{(n+2)}-\nu\sigma_{\theta\theta}^{(n)}-\nu\sigma_{zz}^{(n)} \qquad (6d)$$

$$(2n+1)V^{(n+1)}=\ldots \qquad (6e)$$

$$(2n+1)W^{(n+1)}=\ldots \qquad (6f)$$

$$\sigma_{\theta\theta}^{(n)}=\epsilon\partial V^{(n)}/\partial\theta-\epsilon\partial V^{(n+2)}/\partial\theta+\epsilon U^{(n)}-\epsilon U^{(n+2)}+\nu S_{rr}^{(n)}-\nu S_{rr}^{(n+2)}+\nu\sigma_{zz}^{(n)} \qquad (6g)$$

$$\sigma_{zz}^{(n)}=\ldots \qquad (6h)$$

$$2(1+\nu)\sigma_{\theta z}^{(n)}=\ldots \qquad (6i)$$

In theory, the ninefold infinity of equations are solvable; in practice, however, a truncation scheme must be adopted to effect a manageable solution. The establishment of a truncation scheme requires some care. The natural truncation is to postulate that an Nth-order theory requires satisfaction of the first $N+1$ of each of Eqs. (6) ($n=0,1,\ldots N$). Examination of the equations resulting from such a postulated closure scheme reveals that the $9N+9$ equations contain $9N+27$ coefficient functions, six of which are specified to satisfy boundary conditions on $\eta=\pm1$. Hence, we must either neglect twelve of the higher-order coefficient functions, or we must write twelve additional equations.

To gain insight into what we term "important variables" for an Nth-order theory, we develop the expression for strain energy stored in an element of the cylinder. Using Eqs. (3), we can obtain the result

$$\int_{R-h}^{R+h}W^{*}\bar{r}d\bar{r}d\theta d\bar{z}=\sum_{n=0}^{\infty}E_n(\theta,z)d\theta dz \qquad (7)$$

where W^{*} is the strain energy density. Additional details regarding the manipulations necessary to obtain Eq. (7) may be found in Appendix E of Ref. 5; a similar reduction of the strain energy expression for the generalized plane stress problem appears in Ref. 6. The strain energy expression truncated at $n=N$ is found to contain only the coefficient functions appearing in Table 1—the structure of the truncated expression for strain energy suggests that we postulate that an Nth-order theory contain all coefficient functions appearing in Table 1. That is, an Nth-order approximation will determine only those coefficient functions appearing in the first N terms of the series expansion for the strain energy. From Table 1, we see that the Nth-order theory (as we have now defined such a theory) contains $9N+21$ coefficient functions of which six are known by virtue of known boundary conditions on the surfaces $\eta=\pm1$. As we noted before, writing out the first N equations of each of Eqs. (6), and neglecting all coefficient functions not appearing in Table 1, yields only $9N+9$ equations; thus, we still must obtain six additional equations to obtain a well posed theory.

To obtain these six additional equations necessary to construct the approximate theory (consistent with our hypothesized considerations of important variables), we note that the structure of the field equations (6) suggests considering each equation as defining a particular coefficient function. That is, the first of Eqs. (6) defines $S_{rr}^{(n+1)}$, the second of Eqs. (6) defines $S_{r\theta}^{(n+1)}$, etc. for $n\geq 0$. With this idea, we note that

Table 1 Functions appearing in Nth-order theory

$S_{rr}^{(0)}$	$S_{r\theta}^{(0)}$	$S_{rz}^{(0)}$	$\sigma_{\theta\theta}^{(0)}$	$\sigma_{zz}^{(0)}$	$\sigma_{\theta z}^{(0)}$	$U^{(0)}$	$V^{(0)}$	$W^{(0)}$
$S_{rr}^{(1)}$	$S_{r\theta}^{(1)}$	$S_{rz}^{(1)}$	$\sigma_{\theta\theta}^{(1)}$	$\sigma_{zz}^{(1)}$	$\sigma_{\theta z}^{(0)}$	$U^{(1)}$	$V^{(1)}$	$W^{(1)}$
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
$S_{rr}^{(N-1)}$	$S_{r\theta}^{(N-1)}$	$S_{rz}^{(N-1)}$	$\sigma_{\theta\theta}^{(N-1)}$	$\sigma_{zz}^{(N-1)}$	$\sigma_{\theta z}^{(N-1)}$	$U^{(N-1)}$	$V^{(N-1)}$	$W^{(N-1)}$
$S_{rr}^{(N)}$	$S_{r\theta}^{(N)}$	$S_{rz}^{(N)}$	$\sigma_{\theta\theta}^{(N)}$	$\sigma_{zz}^{(N)}$	$\sigma_{\theta z}^{(N)}$	$U^{(N)}$	$V^{(N)}$	$W^{(N)}$
$S_{rr}^{(N+1)}$	$S_{r\theta}^{(N+1)}$	$S_{rz}^{(N+1)}$				$U^{(N+1)}$	$V^{(N+1)}$	$W^{(N+1)}$
$S_{rr}^{(N+2)}$	$S_{r\theta}^{(N+2)}$	$S_{rz}^{(N+2)}$				$U^{(N+2)}$	$V^{(N+2)}$	$W^{(N+2)}$

the first N equations of each of Eqs. (6) (yielding the original $9N + 9$ equations) will not contain "defining equations" for the six coefficient functions $S_{rr}^{(N+2)}$, $S_{r\theta}^{(N+2)}$, $S_{rz}^{(N+2)}$, $U^{(N+2)}$, $V^{(N+2)}$, and $W^{(N+2)}$. We suggest that the six additional equations sought to properly define our approximate theory may be obtained by writing out the six missing defining equations. We note that in writing these additional equations, all coefficient functions not appearing in Table 1 should be neglected. Our final Nth-order approximate theory then contains $9N + 15$ equations involving $9N + 21$ coefficient functions; six functions are specified to satisfy cylindrical surface boundary conditions.

The boundary conditions on surfaces θ = constant and z = constant are easily derived from the exact three-dimensional elasticity edge conditions; if we note the series representations for each variable and make use of the orthogonality relations for Legendre polynomials, we can show that the three-dimensional boundary conditions imply specification of boundary values on certain of the coefficient functions. Within the context of the approximations regarding series truncation, Table 2 shows precisely the boundary condition that can be satisfied by our Nth-order theory. Note that a first-order theory is capable of satisfying the class of boundary conditions associated with a "shell" theory; a second-order theory has the capability of satisfying additional boundary conditions reflecting an initial St. Venant effect.

We note that a given approximate theory is characterized by the presence of higher-order "correction" terms which cannot be specified on a boundary. For example, Tables 1 and 2 indicate that an Nth-order theory yields $N + 2$ functions associated with the variable u, but only N of these can be specified on a boundary. Thus, the solution only approximately satisfies boundary conditions; the degree of approximation depends on the relative size of these higher-order correction terms. If these correction terms are large over the entire region, one should suspect that a higher-order theory should be utilized.

To illustrate use of the truncation scheme, we consider an axisymmetrically loaded hollow cylinder and assume

$$\partial/\partial\theta = \sigma_{r\theta} = \sigma_{z\theta} = v = 0 \qquad (8)$$

With the aforementioned simplifications, the initial $6N + 6$ field equations for an Nth-order theory (as we have defined such a theory) are given as

$$(2n + 1)S_{rr}^{(n+1)} = -n\epsilon S_{rr}^{(n)} - (n + 1)\epsilon S_{rr}^{(n+2)} - \frac{n\epsilon}{(2n - 1)}\frac{dS_{rz}^{(n-1)}}{dz} - \frac{dS_{rz}^{(n)}}{dz} + \frac{(2n + 1)\epsilon}{(2n - 1)(2n + 3)}\frac{dS_{rz}^{(n+1)}}{dz} + \frac{(n + 1)\epsilon}{(2n + 3)}A(n)\frac{dS_{rz}^{(n+3)}}{dz} + \epsilon\sigma_{\theta\theta}^{(n)} \qquad (9)$$

$$(2n + 1)S_{rz}^{(n+1)} = -n\epsilon S_{rz}^{(n)} - (n + 1)\epsilon S_{rz}^{(n+2)} - \frac{n\epsilon}{2n - 1}\frac{d\sigma_{zz}^{(n-1)}}{dz} - \frac{d\sigma_{zz}^{(n)}}{dz} - \frac{(n + 1)\epsilon}{2n + 3}\frac{d\sigma_{zz}^{(n+1)}}{dz} \qquad (10)$$

$$(2n + 1)U^{(n+1)} = -n\epsilon U^{(n)} - (n + 1)\epsilon U^{(n+2)} + S_{rr}^{(n)} - S_{rr}^{(n+2)} - \nu\sigma_{\theta\theta}^{(n)} - \nu\sigma_{zz}^{(n)} \qquad (11)$$

$$\sigma_{\theta\theta}^{(n)} = \epsilon[U^{(n)} - U^{(n+2)}] + \nu[S_{rr}^{(n)} - S_{rr}^{(n+2)} + \sigma_{zz}^{(n)}] \qquad (12)$$

$$\sigma_{zz}^{(n)} = \frac{n\epsilon}{2n - 1}\frac{dW^{(n-1)}}{dz} + \frac{dW^{(n)}}{dz} - \frac{(2n + 1)\epsilon}{(2n - 1)(2n + 3)}\frac{dW^{(n+1)}}{dz} - \frac{dW^{(n+2)}}{dz} - \frac{(n + 1)\epsilon}{(2n + 3)}A(n) \times \frac{dW^{(n+3)}}{dz} + \nu[S_{rr}^{(n)} - S_{rr}^{(n+2)} + \sigma_{\theta\theta}^{(n)}] \qquad (13)$$

$$(2n + 1)W^{(n+1)} = -n\epsilon W^{(n)} - (n + 1)\epsilon W^{(n+2)} - \frac{n\epsilon}{2n - 1}\frac{dU^{(n-1)}}{dz} - \frac{dU^{(n)}}{dz} + \frac{(2n + 1)\epsilon}{(2n - 1)(2n + 3)}\frac{dU^{(n+1)}}{dz} + \frac{dU^{(n+2)}}{dz} + \frac{(n + 1)\epsilon}{(2n + 3)}A(n)\frac{dU^{(n+3)}}{dz} + 2(1 + \nu)[S_{rz}^{(n)} - S_{rz}^{(n+2)}] \qquad (14)$$

where $n = 0, 1, 2, \ldots N$ and $A(n) = 1, n < N$; $A(N) = 0$.

Table 1 implies that for the axisymmetric problem, there are $6N + 14$ important coefficient functions, four of which are known by virtue of boundary conditions on surfaces $\eta = \pm 1$. To close the Nth-order theory we need to write four additional equations; with our concept of defining equations, we write the four additional equations

$$(2N + 3)S_{rr}^{(N+2)} \cong -(N + 1)\epsilon S_{rr}^{(N+1)} - \frac{(N + 1)\epsilon}{2N + 1}\frac{dS_{rz}^{(N)}}{dz} - \frac{dS_{rz}^{(N+1)}}{dz} + \frac{(2N + 3)\epsilon}{(2N + 1)(2N + 5)}\frac{dS_{rz}^{(N+2)}}{dz} \qquad (15)$$

$$(2N + 3)S_{rz}^{(N+2)} \cong -(N + 1)\epsilon S_{rz}^{(N+1)} - \frac{(N + 1)\epsilon}{(2N + 1)}\frac{d\sigma_{zz}^{(N)}}{dz} \qquad (16)$$

$$(2N + 3)U^{(N+2)} \cong -(N + 1)\epsilon U^{(N+1)} + S_{rr}^{(N+1)} \qquad (17)$$

Table 2 Boundary conditions specified for Nth-order theory

	Surface θ = const		
$u^{(0)}$ or $\sigma_{r\theta}^{(0)}$	$v^{(0)}$ or $\sigma_{\theta\theta}^{(0)}$ $v^{(1)}$ or $\sigma_{\theta\theta}^{(1)}$	$w^{(0)}$ or $\sigma_{\theta z}^{(0)}$ $w^{(1)}$ or $\sigma_{\theta z}^{(1)}$	$N = 1$
$u^{(1)}$ or $\sigma_{r\theta}^{(1)}$	$v^{(2)}$ or $\sigma_{\theta\theta}^{(2)}$	$w^{(2)}$ or $\sigma_{\theta z}^{(2)}$	$N = 2$
$u^{(N-1)}$ or $\sigma_{r\theta}^{(N-1)}$	$v^{(N)}$ or $\sigma_{\theta\theta}^{(N)}$	$w^{(N)}$ or $\sigma_{\theta z}^{(N)}$	$N = N$
	Surface z = const		
$u^{(0)}$ or $\sigma_{rz}^{(0)}$	$w^{(0)}$ or $\sigma_{zz}^{(0)}$ $w^{(1)}$ or $\sigma_{zz}^{(1)}$	$v^{(0)}$ or $\sigma_{\theta z}^{(0)}$ $v^{(1)}$ or $\sigma_{\theta z}^{(1)}$	$N = 1$
$u^{(1)}$ or $\sigma_{rz}^{(1)}$	$w^{(2)}$ or $\sigma_{zz}^{(2)}$	$v^{(2)}$ or $\sigma_{\theta z}^{(2)}$	$N = 2$
$u^{(N-1)}$ or $\sigma_{rz}^{(N-1)}$	$w^{(N)}$ or $\sigma_{zz}^{(N)}$	$v^{(N)}$ or $\sigma_{\theta z}^{(N)}$	$N = N$

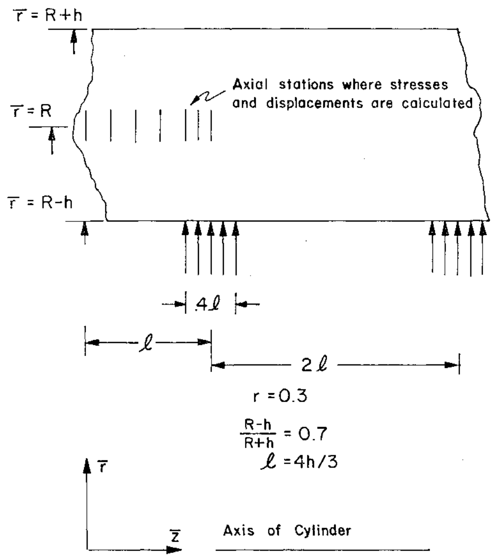

Fig. 2 Cylinder loading.

$$(2N+3)W^{(N+2)} \cong -(N+1)\epsilon W^{(N+1)} - \frac{(N+1)\epsilon}{(2N+1)}\frac{dU^{(N)}}{dz} - \frac{dU^{(N+1)}}{dz} + \frac{(2N+3)\epsilon}{(2N+1)(2N+5)}\frac{dU^{(N+2)}}{dz} + 2(1+\nu)S_{rz}^{(N+1)} \quad (18)$$

Equations (9–18) define the Nth-order theory. Using these equations, first- and second-order approximate theories may be developed. The detailed expansion and subsequent simplification of the field equations is found in Ref. 5. In Ref. 5, we also show that the field equations can be recast into a set of first-order equations suitable for step-by-step integration using a Runge-Kutta scheme.

Comparison with Exact Solution

To substantiate the claim that this method of approach yields improved approximate theories, we use a first- and second-order approximation and compare numerical results with an exact solution. Klosner and Levine[7] generated an exact solution for an infinite cylindrical shell with periodic band loading and used this solution to evaluate different shell theories. We will now add our numerical results to their evaluation. The geometry of the considered problem is shown in Fig. 2. Note that the configuration represents a thick shell; the ratio of mean radius of curvature to thickness is 2.87. Furthermore, the width of the pressure band is

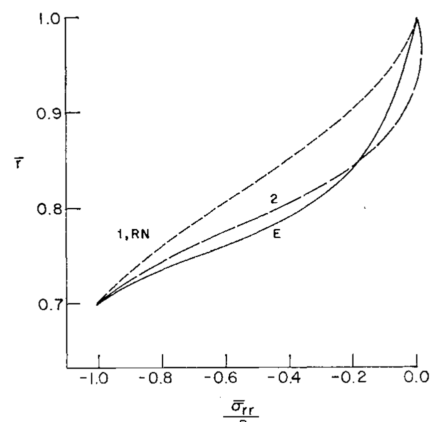

Fig. 3 Transverse normal stress distribution ($\bar{z}/l$ = 1.0).

significantly less than the shell thickness whereas the distance between pressure bands is of the order of the shell thickness. Clearly, this is not a problem where one can comfortably apply a classical shell theory; our intent here is to evaluate our new approximate theories and hopefully gain some insight into the rapidity of convergence of our results.

Equations (9–18) were specialized to the cases $N = 1$ and $N = 2$ and solutions generated for the geometry and loading of Fig. 2. Details of the solution are found in Ref. 5. Typical results are shown in Figs. 3–8 which compare our solutions (denoted by 1 and 2) with the exact solution (E) published by Klosner and Levine. Also included are results obtained in Ref. 7 using a Reissner-Naghdi theory (RN) and results obtained using a shell theory proposed by Zudans (Z) which makes the Kirchoff kinematic hypothesis and neglects transverse shear deformation and transverse normal stress.[8] More complete results are presented in Ref. 5.

Figure 3 presents the distribution of transverse normal stress through the shell wall. Under the pressure band ($\bar{z}/l$

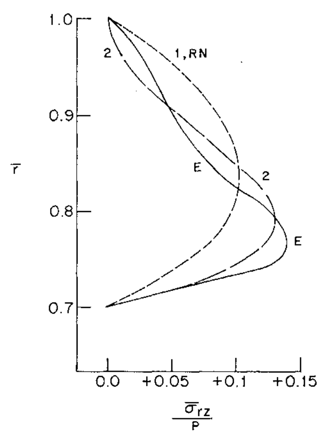

Fig. 4a Shearing stress distribution ($\bar{z}/l$ = 0.6).

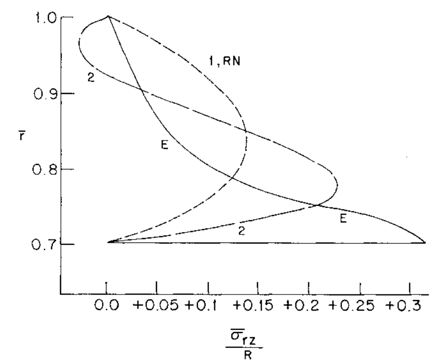

Fig. 4b Shearing stress distribution ($\bar{z}/l$ = 0.8).

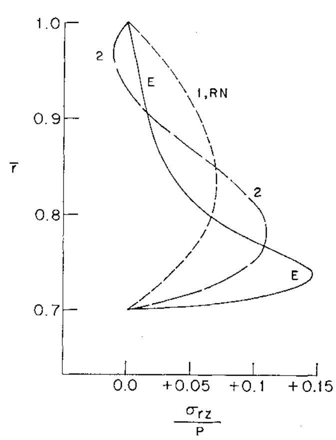

Fig. 4c Shearing stress distribution ($\bar{z}/l$ = 0.9).

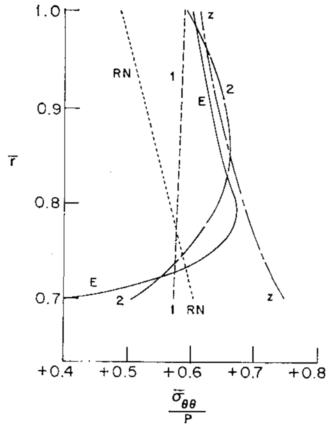

Fig. 5 Circumferential normal stress distribution ($\bar{z}/l$ = 0.9).

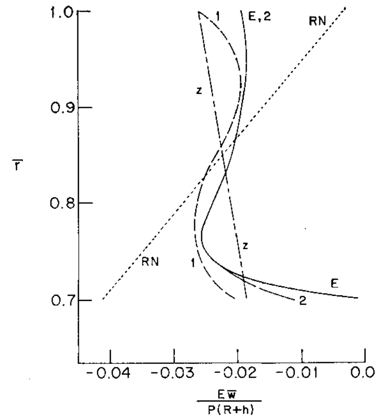

Fig. 7a Distribution of axial displacement ($\bar{z}/l$ = 0.8).

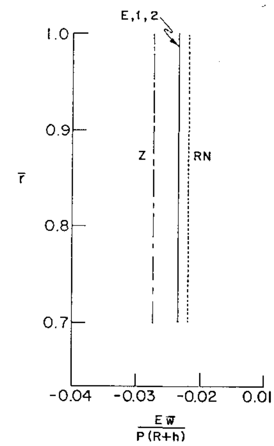

Fig. 7b Distribution of axial displacement ($\bar{z}/l$ = 1.0).

= 1.0), the transverse normal stress is not negligible in comparison to the axial and circumferential normal stresses (in contrast to the assumption made by many classical shell theories). In the region of large transverse normal stress, the new first-order approximation theory 1 coincides with the Reissner-Naghdi (RN) theory whereas the second-order 2 approximation gives improved accuracy—for engineering purposes, all three solutions provide an acceptable representation of the transverse normal stresses. The new second-order theory approximates the exact solution very well and indicates (but, of course, does not prove) that the new approximate solutions will converge to the exact solution as the order of the approximation increased.

Figure 4 depicts the distribution of shearing stress at three axial positions along the shell; the shearing stresses vanish at $\bar{z}/l = 0$ and $\bar{z}/l = 1$. The new first-order theory 1 agrees with the Reissner-Naghdi theory and is a rather poor approximation to the elasticity solution in regions of the shell where the shearing stress is large. The second-order theory represents an improved approximation to the exact solution. At the edge of the pressure band ($\bar{z}/l = 0.8$), the shearing stress, which must vanish at the inner and outer surfaces of the shell, achieves its maximum value very near to the inner surface. To accurately approximate such a distribution, one would expect to require many terms of a Fourier-Legendre series; indeed the new second-order theory 2, while showing an obvious tendency to converge to the elasticity solution and clearly indicating the shift of the maximum value, underestimates the peak value of the shearing stress. The first neglected term of the series is such that it will improve the fit near $\bar{r} = 1$, and shift the peak value toward $\bar{r} = 0.7$.

Figure 5 shows distribution of the circumferential normal stress at a typical position along the shell. Generally, the new first- and second-order theories are better approximations of the elasticity solution than is the Reissner-Naghdi solution.

Figure 6 illustrates axial normal stress distribution at a typical station. As noted in Ref. 7, a very poor approximation to the elasticity solution is provided by the Reissner-Naghdi theory—generally, the predicted stress values do not even exhibit correct polarity. Much improved approximations are provided by the new first- and second-order theories as well as the theory of Zudans.

Figure 7 shows the distribution of axial displacements at typical stations. As was the case for the axial normal stress, the Reissner-Naghdi theory yields a very poor approximation to the elasticity solution. The new first- and second-order approximate solutions as well as Zudans' theory provide considerably better results.

Figure 8 illustrates the radial displacements of the middle surface of the shell as a function of axial position. Both the new first- and second-order approximations agree with the elasticity solution while the Reissner-Naghdi and Zudans' theories provide good approximations. As a matter of interest, the radial displacement of the inner and outer surfaces of the shell as calculated from the new first-order theory is shown. Figures 7 and 8 clearly demonstrate the distortion of the normal element which occurs in this relatively thick shell.

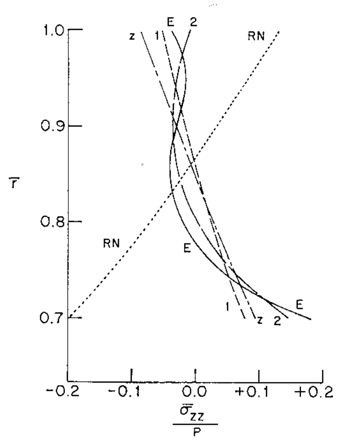

Fig. 6 Axial normal stress distribution ($\bar{z}/l$ = 0.2).

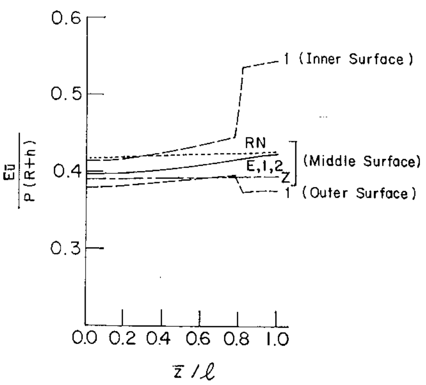

Fig. 8 Radial displacement along axis.

Examination of these typical results indicates that the new first-order approximation (equivalent to a classical shell theory including transverse shear deformation and transverse normal stress) yields significantly improved results when compared with a well-known classical shell theory exhibiting these effects. The new second approximation theory tends to give even better agreement with the exact solution. Note that in Fig. 4, a third-order approximate theory would contribute an additional term tending to improve the comparison near $\bar{r} = 1$ and further shift the maximum value toward $\bar{r} = 0.7$. Figures 7 and 8 clearly show the tendency for the new approximate theories to conform to the exact displacement distributions. The distortion of the normal element (as calculated from the new first order theory) is clearly demonstrated.

Concluding Remarks

We have presented significant ideas which lead toward the development of accurate approximate solutions of certain three-dimensional elasticity problems; in some respects these approximate theories may be considered as higher-order approximate shell theories. Although we have dealt specifically with the cylindrical geometry, extension of the technique to other geometries will incur only algebraic complexities. In the context of an alternate representation of an exact solution to the finite cylinder problem, our approach seems to avoid all problems concerned with convergence and simultaneous satisfaction of boundary conditions which have plagued other investigators. In the context of developing a shell theory, our treatment here avoids all problems concerning treatment of geometric terms of the type $1/(1 + \epsilon\eta)$; only assumptions concerned with defining an Nth-order theory need be introduced.

The success of the method is contingent on the truncation scheme that must be used to effect a useful solution; the comparison with the exact solution seems to validate the applicability of our postulated truncation scheme. Much more work, however, is indicated to further verify the usefulness of a strain energy approach for the definition of a truncation scheme. Examination of our numerical results indicates that relatively few terms of the series solutions may need to be retained to obtain useful engineering results, even in the case of highly localized loadings.

Finally, we note that our investigations provide, in our opinion, an effective complement to direct numerical assaults on the three-dimensional problem in that our reduced two-dimensional equations may offer a new starting point for the generation of exact solutions by numerical means. Thus, we feel that work along these lines merits further pursuit. Since our technique will provide a clear connection between a classical shell theory and higher-order approximations to the exact 3-D solution, extension to other shell geometries is of considerable interest.

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